MATH 596 - INTRODUCTION TO MODULAR FORMS
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ASSIGNMENT 1

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Problem 1

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant meromorphic function.
Show that the set of its periods $L_f := \{\omega \in \mathbb{C}; f(z+\omega) = f(z) \text{ for all } z \in \mathbb{C}\}$ is a discrete subgroup of $\mathbb{C}$.

Problem 2

1. Two lattices $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $L' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ coincide if and only if there exists a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ such that
$$A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}.$$
2. Two lattices L, L' are called *homothetic* if $L = \alpha L'$ for some $\alpha \in \mathbb{C}^\times$. Show that every lattice $L \subset \mathbb{C}$ is homothetic to a lattice of the form $L' = \mathbb{Z} + \mathbb{Z}\tau$ with $\mathrm{Im}(\tau) > 0$.
3. Let $L = \mathbb{Z} + \mathbb{Z}\tau$ and $L' = \mathbb{Z} + \mathbb{Z}\tau'$ with $\mathrm{Im}(\tau), \mathrm{Im}(\tau') > 0$. Show that L and L' are homothetic if and only if there is a matrix
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$
and
$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

Problem 3

Prove the following generalization of Liouville's first theorem: Let f be an entire function, and let L be a lattice in $\mathbb{C}$. Suppose that for any lattice point $\omega \in L$ there is a polynomial P_ω with the property
$$f(z+\omega) = f(z) + P_\omega(z).$$
Then f is a polynomial.

Problem 4

For an odd elliptic function associated with the lattice L the half-lattice points $\omega/2$ for $\omega \in L$, are either zeros or poles.

Problem 5

Show the following recursion formulas for the Eisenstein series G_{2k} for $k \geq 4$:

$$(2k+1)(k-3)(2k-1)G_{2k} = 3\sum_{j=2}^{k-2}(2j-1)(2k-2j-1)G_{2j}G_{2k-2j}.$$

For instance:

$$G_{10} = \frac{5}{11}G_4G_6.$$

In particular, every Eisenstein series is representable as a polynomial in G_4 and G_6 with non-negative coefficients.

Problem 6

Show that any two elliptic functions (for the same lattice L) are algebraically dependent over $\mathbb{C}$.

Problem 7

Let $L \subset \mathbb{C}$ be a lattice. Show that there is no elliptic function $f \in K(L)$, with $\mathbb{C}(f) = K(L)$.