MATH 596 - INTRODUCTION TO MODULAR FORMS
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ASSIGNMENT 10

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Problem 1

Let $k \geq 4$ and $m \geq 1$ be integers and let

$$\Gamma_\infty^J := \left\{ \left(\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) \;\middle|\; n, \mu \in \mathbb{Z} \right\}$$

be the stabilizer of infinity in the Jacobi group $\Gamma^J := \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. We define the Eisenstein series of weight k and index m to be

$$E_{k,m}(\tau, z) := \sum_{\gamma \in \Gamma_\infty^J \backslash \Gamma^J} 1 \mid_{k,m} \gamma.$$

1. Show that the series converges normally and is invariant under the action $|_{k,m}$ of G^J. Note that if k is odd, then $E_{k,m} = 0$ and thus we assume from now on that k is even.
2. Show that we have

$$E_{k,m}(\tau, z) = \frac{1}{2} \sum_{(c,d)=1} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{-k} e\left(\frac{-mcz^2}{c\tau + d} + m\lambda^2 \frac{a\tau + b}{c\tau + d} + 2m\lambda \frac{z}{c\tau + d} \right).$$

   Here, the first sum runs over pairs of coprime integers $c, d \in \mathbb{Z}$. (First, work out a system of representatives for $\Gamma_\infty^J \backslash \Gamma^J$ and don't forget that the group Γ^J is not a direct product.)
3. (* This is a bit hard) Let $e_{k,m}(n, r)$ be the Fourier coefficient of index (n, r) of $E_{k,m}$. Show that if $4nm - r^2 < 0$, then $e_{k,m}(n, r) = 0$ and if $4nm - r^2 = 0$ with $r \equiv 0 \bmod (2m)$, then $e_{k,m}(n, r) = 1$. This shows that $0 \neq E_{k,m} \in J_{k,m}$ (and it is not a cusp form).

   Hint: Separate the sum over $c = 0$ and $c \neq 0$. The first one gives the "constant term(s)". Write the second part as

$$\sum_{c=1}^{\infty} c^{-k} \sum_{\substack{d \bmod c \\ (d,c)=1}} \sum_{\lambda \bmod c} e\left(m \frac{\bar{d}\lambda^2}{c} \right) F_{k,m}(\tau + d/c, z - \lambda/c),$$

   where

$$F_{k,m}(\tau, z) := \sum_{r,s \in \mathbb{Z}} (\tau + r)^{-k} e\left(-m \frac{(z+s)^2}{\tau + r} \right)$$

and $\bar{d}$ denotes the inverse of d modulo c. Note that this function is 1-periodic in τ and z. Show that

$$F_{k,m}(\tau, z) = \sum_{r,n\in\mathbb{Z}} I(n,r) q^n \zeta^r,$$

where

$$I(n,r) = \int_{\mathrm{Im}(\tau)=C_1} \tau^{-k} e(-n\tau) \int_{\mathrm{Im}(z)=C_2} e(-mz^2/\tau - rz)\, dz\, d\tau$$

with $C_1, C_2 > 0$ arbitrary. (The integrals are line integrals over the line defined by $\mathrm{Im}(\tau) = C_1$ and $\mathrm{Im}(z) = C_2$, respectively. You can show this identity using the Poisson summation formula.)

Then show that $I(n,r) = 0$ if $4nm - r^2 = 0$.

4. Recall that a holomorphic Jacobi form is called a cusp form if the Fourier coefficients of index (n,r) with $4nm - r^2 = 0$ vanish (note that this is equivalent to saying that the corresponding vector valued modular form is a cusp form). Show that the first Jacobi cusp forms of index 1 occur in weight 10 and 12 and are given explicitly as $\phi_{10,1} := \frac{1}{144}(E_6 E_{4,1} - E_4 E_{6,1})$ and $\phi_{12,1} := \frac{1}{144}(E_4^2 E_{4,1} - E_6 E_{6,1})$. Here, E_4 and E_6 are the Eisenstein series of weight 4 and 6 for $\mathrm{SL}_2(\mathbb{Z})$, respectively, normalized such that the constant coefficient in the Fourier expansion is equal to 1.

Problem 2

Let L_k be the differential operator defined in class:

$$L_k = 8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2} - \frac{2k-1}{z}\frac{\partial}{\partial z}.$$

1. Show that
$$L_k(\phi\mid_{k,m} \gamma) = (L_k\phi)\mid_{k+2,m} \gamma$$
for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.
2. We have that L_k acts on the space $A^+_{k,m}$ defined in class in the context of Theorem 8 (even formal power series whose coefficients satisfy the recursion relation under the action of $\gamma \in \mathrm{SL}_2(\mathbb{Z})$) and maps $A^+_{k,m}$ into $A^+_{k+2,m}$.