MATH 596 - INTRODUCTION TO MODULAR FORMS
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ASSIGNMENT 11

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Problem 1

Let $k \in \mathbb{Z}$ and let $f \in M_k^!$ be a weakly holomorphic modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Show that the Fourier coefficients $c_f(n)$ of satisfy the following estimate: There are $C_1, C_2 > 0$, such that we have for all $n \in \mathbb{Z}$:

$$c_f(n) \leq C_1 e^{C_2 \sqrt{n}}.$$

Problem 2

In this problem we apply the ideal of a regularized "inner product" to the product of Eisenstein series. Let G_k be the Eisenstein series of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Show that

$$\lim_{T \to \infty} \int_{\mathcal{F}_T} G_k(\tau) \overline{G_k(\tau)} v^{k-s} \, \frac{dudv}{v^2}$$

exists for large enough $\mathrm{Re}(s)$ and that it can be continued to a meromorphic function in s that extends at least to a half-plane $\mathrm{Re}(s) > -c$ for $c > 0$. We define (G_k, G_k) to be the constant term in the Laurent expansion at $s = 0$ of this continuation. Show that

$$(G_k, G_k) = \lim_{T \to \infty} \left(\int_{\mathcal{F}_T} G_k(\tau) \overline{G_k(\tau)} v^k \, \frac{dudv}{v^2} - \frac{B_k^2}{4k^2(k-1)} T^{k-1} \right)$$

Remark: It can be shown that in fact

$$(G_k, G_k) = (-1)^{k/2-1} \frac{(k-1)!(k-2)!}{2^{3k-3}\pi^{2k-1}} \zeta(k) \zeta(k-1).$$

(So, in particular it is nonzero but not positive in general! For a proof, see the paper "The Rankin-Selberg method for automorphic functions that are not of rapid decay" by D. Zagier. If you look at it, you find the result on page 435 and you can also see on p. 434 that our regularization is equivalent to the one used by Zagier.)

Problem 3

Let $K \subset V$ be an even, positive definite lattice of rank n in a rational quadratic space $(V = K \otimes_{\mathbb{Z}} \mathbb{Q}, Q)$. Let $\mu \in V(\mathbb{R})$ and $\alpha, \beta \in \mathbb{Z}$. For $h \in K'/K$, we let

$$\Theta_{K+h}(\tau, \mu, \alpha, \beta) := \sum_{\lambda \in K+h} e\left(Q(\lambda + \beta\mu)\tau - (\lambda + \beta\mu/2, \alpha\mu)\right)$$

and define the vector-valued theta function

$$\Theta_K(\tau, \mu, \alpha, \beta) = \sum_{h \in K'/K} \Theta_{K+h}(\tau, \mu, \alpha, \beta) \mathfrak{e}_h.$$

(Here, $e(x) = e^{2\pi i x}$, as usual.) Show that

$$\Theta_K(\gamma\tau, \mu, a\alpha + b\beta, c\alpha + d\beta) = \phi(\tau)^n \rho_K(\gamma, \phi)\Theta_K(\tau, \mu, \alpha, \beta)$$

for all $(\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and where ρ_K is the Weil representation attached to K.