MATH 596 - INTRODUCTION TO MODULAR FORMS
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ASSIGNMENT 2

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Problem 1

Let $L \subset \mathbb{C}$ be a lattice with the property $g_2(L) = 8$ and $g_3(L) = 0$. The point $(2, 4)$ lies on the affine elliptic curve $y^2 = 4x^3 - 8x$. Let $+$ be the addition (for points on the corresponding projective curve). Show that $2 \cdot (2, 4) := (2, 4) + (2, 4)$ is the point $(9/4, -21/4)$.

Problem 2

(Corrected on Jan 20) Let L be a lattice and

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)},$$

where σ is the function

$$\sigma(z) = z \prod_{\omega \in L \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)$$

which can also be used to prove Abel's theorem.

The function ζ is called *Weierstrass' ζ-function*, which is probably a bit less intruiging than the function also denoted ζ and attributed to Riemann...

1. First show that the infinite product defining σ converges normally and thus defines an entire function and has a simple zero at all lattice points.
2. Show that $\zeta'(z) = -\wp(z)$. Moreover, it satisfies a transformation law of the same form as the function we used to prove Abel's theorem in class, i.e.

$$\sigma(z + \omega) = e^{a_\omega z + b_\omega} \sigma(z).$$

What is the relation to $\vartheta(\tau, z)$?

4. Assume that $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\operatorname{Im}(\omega_2/\omega_1) > 0$ and extend the definition of ζ to the lattice L. Let $\eta_1 = \zeta(z + \omega_1) - \zeta(z)$ and $\eta_2 = \zeta(z + \omega_2) - \zeta(z)$. Show *Legendre's relation*:

$$\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i.$$

Problem 3

Show that the series

$$\xi(z) = \frac{-1}{z} - \sum_{\omega \in L} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right)$$

defines an odd primitive of $\wp$ and we have in fact $\xi(z) = -\zeta(z)$.

Problem 4

Let $f : U \to \mathbb{C}$ be a non-constant meromorphic function on an open connected subset $U \subset \mathbb{C}$ which satisfies the differential equation

$$f'^2 = 4f^3 - g_2 f - g_3,$$

where $g_2 = g_2(L) = 60G_4(L)$ and $g_3 = g_3(L) = 140G_6(L)$. Show that $f(z) = \wp(z+a)$ for some $a \in \mathbb{C}$.

Problem 5

Let $L \subset \mathbb{C}$ be a lattice and show that $\Delta(L) := g_2(L)^3 - 27g_3(L)^2 \neq 0$. (Hint: $\Delta(L)$ is the discriminant of the cubic polynomial $4x^2 - g_2 x - g_3$.)