MATH 596 - INTRODUCTION TO MODULAR FORMS
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ASSIGNMENT 3

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Problem 1

Show that the *Weil reciprocity law holds*: If $f \in K(L)$ is an elliptic function and $D = \sum_{z \in \mathbb{C}/L} n_z \cdot (z)$ is a divisor on $\mathbb{C}/L$, we define

$$f(D) := \prod_{z \in \mathbb{C}/L} f(z)^{n_z} \in \mathbb{C},$$

whenever D does not contain any point in the divisor of f. Show that if f and g are nonzero elliptic functions with no common points in the divisor (i.e. $\operatorname{ord}_z(f) \neq 0 \Rightarrow \operatorname{ord}_z(g) = 0$ and vice versa) then

$$f(\operatorname{div}(g)) = g(\operatorname{div}(f)).$$

Problem 2

1. Convince yourself that the map
$$\mathbb{H} \times \mathrm{SL}_2(\mathbb{R}) \to \mathbb{H} \quad (\tau, \gamma) \mapsto \gamma\tau := \frac{a\tau + b}{c\tau + d}$$
for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defines a group action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$.
2. Show that this action is transitive.
Hint: Show that for every $\tau \in \mathbb{H}$ there is a matrix $\gamma_\tau \in \mathrm{SL}_2(\mathbb{R})$, s.t. $\gamma_\tau i = \tau$.
3. Show that the sabilizer of the point $i \in \mathbb{H}$ is equal to
$$\mathrm{SO}_2(\mathbb{R}) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \ \mid \ \gamma^T\gamma = \mathrm{I}_2\},$$
where I_2 is the 2×2 identity matrix. Conclude that the map
$$\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \to \mathbb{H} \quad \gamma\mathrm{SO}_2(\mathbb{R}) \mapsto \gamma i$$
is a bijection.

Problem 3

1. Show that for $k \geq 4$ even the Eisenstein series
$$G_k(\tau) = \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(c\tau + d)^k}$$

defines a holomorphic function on $\mathbb{H}$ and satisfies

$$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k G_k(\tau)$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

2. Let $\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\}$, i.e. the subgroup of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ generated by $\pm T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Note that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, then the pair $(c,d) \in \mathbb{Z}^2$ only depends on γ modulo Γ_∞, acting on the left, i.e. only on the class $[\gamma] := \Gamma_\infty \gamma \in \Gamma_\infty \backslash \Gamma$.
3. Show that
$$G_k(\tau) = 2\zeta(k) \sum_{[\gamma]\in\Gamma_\infty\backslash\Gamma} j(\gamma,\tau)^{-k},$$
where $j(\gamma,\tau) = c\tau + d$ if γ has bottom row (c,d) as above and $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ is the Riemann zeta function.

Problem 4

Show that the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is generated by the two matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Problem 5

(This exercise is optional/bonus.)
Let $D < 0$ be a discriminant, i.e. $D \equiv 0, 1 \pmod 4$. Let Q_D denote the set of integral positive definite binary quadratic forms of discriminant $D < 0$, i.e. Q_D is the set of all $Q(x,y) = ax^2 + bxy + cy^2$ with $a,b,c \in \mathbb{Z}$, $D = b^2 - 4ac$ and $a > 0$. We briefly write $Q = [a,b,c]$ for this quadratic form.

1. Show that the map
$$Q_D \times \Gamma \to Q_D, \quad (Q,\gamma) \mapsto Q(ax+by, cx+dy) \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
defines a group action of Γ on the set Q_D.
2. Denote by $\mathcal{Q}_D$ the set of orbits under this group action and $h(D) = \#\mathcal{Q}_D$. Show that $h(D)$ is finite.
   Hint: For $Q = [a,b,c]$, consider the point
$$\alpha_Q = \frac{-b+\sqrt{D}}{2a},$$
which is the unique root of $Q(\tau,1)$ in the complex upper half-plane $\mathbb{H}$.