MATH 596 - INTRODUCTION TO MODULAR FORMS
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ASSIGNMENT 5

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Problem 1

Recall the definition of the Bernoulli numbers B_m as coefficients of the Taylor expansion

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

1. Show that $B_m \in \mathbb{Q}$ and $B_{2m+1} = 0$ for all $m \in \mathbb{N}$.
2. We also have
$$\frac{z}{e^z-1} + \frac{z}{2} = \frac{z}{2}\frac{e^z+1}{e^z-1} = 1 + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}$$
3. Show the expansion
$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{m=1}^{\infty}\left(\frac{1}{z+m} + \frac{1}{z-m}\right) = \frac{1}{z} + \sum_{0\neq n\in\mathbb{Z}}\left(\frac{1}{z-n} + \frac{1}{n}\right).$$
Hint: Consider the function
$$f(w) = \frac{z}{w(z-w)}\pi\cot(\pi w)$$
(for fixed $z \in \mathbb{C}\setminus\mathbb{Z}$). Show that f has only poles of order 1 and 2 and is holomorphic outside of $\mathbb{Z}\cup\{z\}$. Then consider a suitable square Q in $\mathbb{C}$ and calculate the integral of f over the boundary of Q.
4. Show the formula (for $k \in \mathbb{N}$)
$$\zeta(2k) = \frac{(2\pi)^{2k}(-1)^{k+1}B_{2k}}{2\cdot(2k)!}.$$
Hint: Show that
$$z\cot(z) = 1 - \sum_{k=1}^{\infty}(-1)^{k+1}B_{2k}\frac{2^{2k}z^{2k}}{(2k)!} = 1 - 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{z^{2k}}{n^{2k}\pi^{2k}}.$$

Problem 2

Show the Hurwitz identity:

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{r,s=n}\sigma_3(r)\sigma_3(s).$$

Problem 3

Let $f \in S_k$ be a cusp form with Fourier expansion

$$\sum_{n=1}^{\infty} a_n q^n.$$

Show that there is a constant $C > 0$, such that

$$|a_n| \leq C n^{k/2}.$$

Hint: Consider the function $F(\tau) = v^{k/2} f(\tau)$ (with $v = \mathrm{Im}(\tau)$), which is rapidly decreasing for $v \to \infty$ to show that $|f(\tau)| \leq c \cdot v^{-k/2}$ for some $c > 0$.