MATH 596 - INTRODUCTION TO MODULAR FORMS
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ASSIGNMENT 6 - HECKE OPERATORS

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First recall the definition of the operator $T(n)$ acting on the set $\mathcal{R}$ of lattices in $\mathbb{C}$ from class. It is completely determined by ($n \in \mathbb{N}$)

$$T(n)L = \sum_{\substack{L' \subset L \text{ sublattice} \\ [L:L']=n}} L'$$

for $L \in \mathcal{R}$.

Problem 1

Show that the sum defining $T(n)L$ is finite. More precisely, the number of sublattices of L with index n is equal to the number of subgroups of $(\mathbb{Z}/n\mathbb{Z})^2$ of order n.

Problem 2

Show that the operators $T(n)$ satisfy the following relations:

1. $T(m)T(n) = T(mn)$ if $\gcd(m, n) = 1$
2. $T(p^n)T(p) = T(p^{n+1}) + pT(p^{n-1})R_p$, where R_p is the homothety operator $R_pL = pL$.
3. $T(m)T(n) = T(n)T(m)$ for all $m, n \in \mathbb{N}$.

Problem 3

1. Recall that there is a bijection between functions on $\mathcal{R}$ with $F(\lambda L) = \lambda^{-k}F(L)$ for $\lambda \in \mathbb{C}^\times$ and functions on $\mathbb{H}$ that satisfy $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. This bijection is realized by setting
$$f(\omega_1/\omega_2) = \omega_2^k F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$$
for $\omega_1/\omega_2 \in \mathbb{H}$.
2. Now let T_n be the Hecke operator acting on functions $f : \mathbb{H} \to \mathbb{C}$ satisfying $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for defined by $(T_nf)(\tau) = G(\mathbb{Z}\tau + \mathbb{Z})$, where G is the function on lattices obtained as
$$G = n^{k-1}T(n)F$$
and F is the function on lattices corresponding to f as above. Show that
$$(T(n)f)(\tau) = n^{k-1} \sum_{\substack{a \in \mathbb{N} \\ ad=n \\ 0 \le b < d}} d^{-k} f\left(\frac{a\tau + b}{d}\right).$$

Hint: Let

$$M_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \ \middle| \ a \geq 1, ad = n, 0 \leq b < d \right\}$$

and define ML for $M \in M_n$ and $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ as the lattice spanned by the entries of $M \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$. Show that the map

$$M_n \to \{L' \subset L \ \mid L' \text{ sublattice with } [L : L'] = n\}, \quad M \mapsto ML$$

is a bijection. (Also note that this is a way to prove the finiteness in Problem 1).

Problem 4

Show that the Hecke operators are self-adjoint (hermitian) operators on S_k with respect to the Petersson inner product, that is

$$\langle T(n)f, g\rangle = \langle f, T(n)g\rangle.$$