MATH 596 - INTRODUCTION TO MODULAR FORMS
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ASSIGNMENT 7

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Problem 1

1. Let N be a positive integer and define
$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; a \equiv d \equiv 1 \bmod N, b \equiv c \equiv 0 \bmod N \right\}.$$
Show that $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and that there is an isomorphism
$$\Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$
Hint: The main point is to show that the reduction modulo N is surjective. For this, first show that there are integers c', d' with $\gcd(c', d') = 1$ and $c' \equiv c \bmod N$ and $d' \equiv d \bmod N$. Next show that there are integers a', b', such that $a \equiv a' \bmod N$ and $b \equiv b' \bmod N$, such that
$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

2. Show that
$$|\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{p \mid N} (1 - \frac{1}{p^2}),$$
where the product is over all prime divisors of N. Hint: first consider $N = p^m$ for a prime p and $m \geq 1$.

Problem 2

Let
$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N, a \equiv d \equiv 1 \bmod N \right\}$$
and
$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}.$$
Both, $\Gamma_0(N)$ and $\Gamma_1(N) \subset \Gamma_0(N)$ are subgroups of $\mathrm{SL}_2(\mathbb{Z})$ containing $\Gamma(N)$. Such groups are called *congruence subgroups* of the modular group.

1. Show that
$$\Gamma_1(N) \to \mathbb{Z}/N\mathbb{Z}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b$$
is surjective and has kernel $\Gamma(N)$.

2. Show that
$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d$$
is surjective and has kernel $\Gamma_1(N)$.
3. Show that the index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ is given by
$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p \mid N} (1 + \frac{1}{p})$$

Problem 3

1. Fill in the details for the proof that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by
$$S := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right) \quad \text{and} \quad T := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right)$$
with relations
$$S^2 = (ST)^3 = Z = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right)$$
and $S^8 = 1$, i.e. $\langle S, T \mid S^2 = (ST)^3, S^8 \rangle$ is a presentation of $\mathrm{Mp}_2(\mathbb{Z})$ (i.e. it is the free group generated by S and T modulo the relations given above).
2. Show that $\mathrm{Mp}_2(\mathbb{Z})$ is, up to isomorphy, the only nontrivial central extension of $\mathrm{SL}_2(\mathbb{Z})$ by a group of order 2. What this means is that if there is a short exact sequence of groups
$$1 \to \{\pm 1\} \to G \to \mathrm{SL}_2(\mathbb{Z}) \to 1,$$
for a group G, then $G \cong \mathrm{Mp}_2(\mathbb{Z})$ or $G \cong \mathrm{SL}_2(\mathbb{Z}) \times \{\pm 1\}$.
3. Let $C = \{\chi : \mathrm{Mp}_2(\mathbb{Z}) \to \mathbb{C}^\times \mid \chi \text{ is a homomorphism}\}$ be the group of (linear) characters of $\mathrm{Mp}_2(\mathbb{Z})$. Show that C is cyclic of order 24 and generated by the character
$$\varepsilon(T) = e^{2\pi i/24}, \quad \varepsilon(S) = e^{-2\pi i/8}.$$