MATH 596 - INTRODUCTION TO MODULAR FORMS
WINTER 2016
ASSIGNMENT 8

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We write throughout $e(x) = e^{2\pi i x}$ and $q = e(\tau)$.

Problem 1

Show that the Eisenstein series $G_k \in M_k$ are *Hecke eigenforms*, i.e. $T_n G_k = \lambda_n G_k$ for all n with $\lambda_n \in \mathbb{C}$.

Problem 2

Let A be a finite abelian group and $Q : A \to \mathbb{Q}/\mathbb{Z}$ be a quadratic form on A, i.e. $Q(xa) = x^2 Q(a)$ for all $x \in \mathbb{Z}$ and $a \in A$ and the map $(\cdot,\cdot) : A \times A \to \mathbb{Q}/\mathbb{Z}$ defined by $(x, y) = Q(x + y) - Q(x) - Q(y)$ is bilinear and non-degenerate, i.e. $(x, y) = 0$ for all $y \in A$ implies that $x = 0$. (We call such a pair (A, Q) a *finite quadratic module.*)

1. Let S and T be the two generators of $\mathrm{Mp}_2(\mathbb{Z})$ given by
$$S := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right) \quad \text{and} \quad T := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right).$$
We denote by s and t the linear operators on $\mathbb{C}[A]$, such that
$$t\mathfrak{e}_x = e(Q(x))\mathfrak{e}_x$$
and
$$s\mathfrak{e}_x = \frac{1}{\sqrt{|A|}} \sum_{y \in A} \mathfrak{e}_y e(-(x, y))$$
for a constant $\sigma \in \mathbb{C}$. Show that if we let $\rho(T) = t$ and $\rho(S) = \sigma s$ for a constant $\sigma \in \mathbb{C}$, then this defines a representation of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[A]$ if and only if
$$\sigma = \frac{1}{\sqrt{|A|}} \sum_{x \in A} e(-Q(x))$$
and σ has to be an 8-th root of unity (we will see that this holds later in class).
2. Compute the constant σ in the case that $A = L'/L$ for $L = \mathbb{Z}^n$ with quadratic form $Q(x) = \sum_{i=1}^n x_i^2$ (and confirm that it is an 8-th root of unity).

Problem 3

Let L be an even lattice and let $S = ((\lambda_i, \lambda_j))$ be the Gram matrix of a basis of L. Show that
$$|\det(S)| = |L'/L|.$$

Problem 4

Let (V, Q) be a quadratic space over $\mathbb{R}$ of type (a, b) and let $v_1, \ldots, v_n$ $(n = a+b)$ be a basis of V, such that $|(v_i, v_j)| = \delta_{i,j}$. Let Λ be the lattice spanned by these vectors. Recall that we normalized the measure on V so that the volume of V/Λ is one.

1. Show that if $b = 0$, then the Fourier transform of $e^{-2\pi Q(x)}$ is $e^{-2\pi Q(x)}$.
2. Let $v \in V$. The Fourier transform of $f(x+v)$ is equal to $e((x, v))\hat{f}(x)$.
3. The Fourier transform of $f(ax)$ is equal to $a^{-n}\hat{f}(x/a)$ for $a > 0$.
4. Prove Theorem 20, the theta transformation formula: If L is an even, positive definite lattice of rank n, then

$$\Theta_{L+\mu}(-1/\tau) = \frac{e(-n/8)}{\sqrt{|L'/L|}} \sum_{\nu \in L'/L} e(-(\mu, \nu))\Theta_{L+\nu}.$$

Problem 5

Let $n = 4n'$ for a positive integer n' and $V = \mathbb{Q}^n$ with quadratic form

$$Q(x) = \frac{1}{2}\sum_{i=1}^{n} x_i^2,$$

so that $(x, y) = \sum_{i=1}^{n} x_i y_i$.

Let $\Lambda_0 = \mathbb{Z}^n$ and $\Lambda_1 = \{x \in \Lambda_0 \mid (x, x) \equiv 0 \bmod 2\}$.

1. Λ_1 is a sublattice of Λ_0 of index 2.
2. Let $E_n \subset V$ be the submodule of V generated by Λ_1 and $(\frac{1}{2}, \ldots, \frac{1}{2})$. Show that

$$E_n = \left\{ x \in \mathbb{Z}^n \cup (\mathbb{Z} + \frac{1}{2})^n \mid \sum_{i=1}^{n} x_i \equiv 0 \bmod 2 \right\}.$$

3. Show that if n' is even (i.e. $n \equiv 0 \bmod 8$), then $E_n' = E_n$ (i.e. E_n is *unimodular*) and the theta function of E_n is a modular form (scalar valued) for $\mathrm{SL}_2(\mathbb{Z})$.
4. We say that two lattices (L, Q) and (M, Q') are isomorphic (or isometric) if there is an isomorphism of $\mathbb{Z}$-modules $\phi : L \to M$, such that $Q'(\phi(x)) = Q(x)$. Show that if L and M are isomorphic lattices, then $\Theta_L(\tau) = \Theta_M(\tau)$.
5. Show that $L_1 = E_8 \oplus E_8$ is not isomorphic to $L_2 = E_{16}$ but the theta functions of L_1 and L_2 agree. Determine the theta functions of E_8, L_1 and L_2 explicitly.
6. Let L be an even unimodular lattice of rank r. The theta function for L can be written as

$$\Theta_L(\tau) = \sum_{n=0}^{\infty} r_L(n) q^n,$$

where $r_L(n)$ is the representation number

$$r_L(n) = \{\lambda \in L \mid Q(\lambda) = n\}.$$

Show that there is a constant, such that
$$r_L(n) \leq Cn^{r/2-1}.$$

Problem 6

Let
$$\vartheta_1(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2},$$
$$\vartheta_2(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2},$$
and
$$\vartheta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2/2}.$$

1. Show that
$$\Delta(\tau) = 2^{-8}(\vartheta_1(\tau)\vartheta_2(\tau)\vartheta_3(\tau))^8.$$

2. Show that
$$\Delta(\tau) = q\left(\sum_{n\in\mathbb{Z}} (-1)^n q^{(3n^2+n)/2}\right)^{24}.$$
Numbers of the form $(3n^2+n)/2$ are called *pentagonal* numbers.