MATH 596 - INTRODUCTION TO MODULAR FORMS
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ASSIGNMENT 9

DR. STEPHAN EHLEN

Problem 1

Let $V = M_2^0(\mathbb{Q})$ be the space of rational 2×2 matrices with trace 0. It is a quadratic space together with the quadratic form $Q(X) = -N \det(X)$.

1. The type of (V, Q) is $(2, 1)$.
2. Consider the lattice
$$L := \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} \ \mid \ a, b, c \in \mathbb{Z} \right\} \subset V.$$
Show that L is even and $L'/L \cong \mathbb{Z}/2N\mathbb{Z}$ with (finite) quadratic form
$$Q(x) = \frac{x^2}{4N} \mod \mathbb{Z}.$$
3. Show that the action of $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ on V via conjugation is an isometry.
4. Show that under this action, the group $\Gamma_0(N)$ preserves the lattice L.
5. Show that the map
$$\mathbb{H} \to \mathbb{D} = \{z \subset V(\mathbb{R}) \ \mid \ \dim(z) = 2, z \text{ positive definite } \}$$
defined by
$$\tau \mapsto \mathbb{R}\mathrm{Re}(Z) \oplus \mathbb{R}\mathrm{Im}(Z)$$
with
$$Z = \begin{pmatrix} \tau & -\tau^2 \\ 1 & -\tau \end{pmatrix}$$
is a bijection (first check that the definition makes sense).
6. Show that under this identification, the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{D}$ corresponds to the usual action via linear fractional transformations on $\mathbb{H}$.
7. * Show that the Siegel theta function attached to L is real analytic in $z \in \mathbb{D} \cong \mathbb{H}$ (as a function on the complex upper half-plane).

Problem 2

In this problem we introduce modular forms on subgroups of the modular group which we didn't have in class (explicitly). But we need them to deal properly with sums of squares as we shall see below. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index. Recall the action of $\mathrm{SL}_2(\mathbb{Z})$ on functions $f : \mathbb{H} \to \mathbb{C}$ via $(\gamma, f) \mapsto f \mid_k \gamma$, where $f \mid_k \gamma(\tau) = f(\gamma\tau)(c\tau + d)^{-k}$ for γ with bottom row (c, d).

We call a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ a holomorphic modular form of weight $k \in \mathbb{Z}$ for Γ if

(1) $f \mid_k \gamma = f$ for all $\gamma \in \Gamma$
(2) and f is holomorphic at the cusps of Γ.

The second condition means the following: The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{P}^1(\mathbb{Q})$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x : y) = (ax + by : cx + dy)$. The equivalence classes under this action (restricted to Γ) are called the cusps of Γ. The group $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$ and we say that f is holomorphic at a cusp $[(\alpha : \beta)]$ of Γ if $f \mid_k \gamma$ is holomorphic at ∞, where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is any matrix, such that $(\alpha : \beta) = \gamma\infty$ (here, $\infty = (1 : 0)$). We call f a *cusp form* if for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ the constant coefficient of the Fourier expansion of $f \mid_k \gamma$ vanishes. (Note that $f \mid_k \gamma$ will have a Fourier expansion since it is periodic of period N_γ, where N_γ is the smallest power of T, such that $T^{N_\gamma} \in \gamma^{-1}\Gamma\gamma$. The order of vanishing at a cusp $[(\alpha : \beta)]$ of Γ is then defined to be the smallest index n, s.t. $a_n \neq 0$ in the expansion $(f \mid_k \gamma)(\tau) = \sum_n a_n e(n\tau/N_\gamma)$ (if $\gamma\infty = (\alpha : \beta)$).)

1. Show that indeed $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$.
2. Show that the notion "cusp form" is well defined (i.e. vanishing at all cusps does not depend on the choice of γ or $(\alpha : \beta)$ as a representative of the cusp $[(\alpha : \beta)]$ above). Even more, the order of vanishing at a cusp does also not depend on these choices.
3. Show that if f is a holomorphic modular form for Γ, then
$$\sum_{\tau \in \Gamma\backslash\mathbb{H}} \frac{\mathrm{ord}_\tau(f)}{e_\Gamma(\tau)} + \sum_{c \in C(\Gamma)} \mathrm{ord}_c(f) = \frac{k[\mathrm{PSL}_2(\mathbb{Z}) : \bar{\Gamma}]}{12}.$$
Here, $\bar{\Gamma}$ is the image of Γ in $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$, $e_\Gamma(\tau)$ is the order of the stabilizer of τ in $\bar{\Gamma}$, $C(\Gamma)$ is the set of cusps of Γ and $\mathrm{ord}_c(f)$ is the order of vanishing of $f \mid_k \gamma$ at ∞ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ as above with $\gamma\infty = c$.
4. Let G_4 be the Eisenstein series of weight 4 for $\mathrm{SL}_2(\mathbb{Z})$. Show that the space of modular forms of weight 4 for the group $\Gamma_0(4)$ is 3-dimensional and generated by $G_4(\tau)$, $G_4(2\tau)$ and $G_4(4\tau)$.
5. Show that the number of ways any integer n can be represented as a sum of eight squares
$$r_8(n) = \#\{(x_i)_i \in \mathbb{Z}^8 \mid \sum_{i=1}^{8} x_i^2 = n\}$$
is equal to
$$r_8(n) = 16 \sum_{d \mid n} (-1)^{n-d} d^3.$$